Advanced Quantum Field Theory (2024/2025)
TP 3 - The Riemann functional equation

In this exercise we will prove the famous reflection formula of the Riemann zeta function

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)\,, \tag{1}$$

where $\Gamma(s)$ is the Euler Gamma function and $\zeta(s)$ the Riemann zeta function. The zeta function and this formula in particular, that was proved by B. Riemann in his famous paper from 1859 *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (*On the Number of Primes Less Than a Given Magnitude*), find ample applications in physics. In the lecture, we will see this formula being used in the context of the Casimir effect, and the zeta function more generally as a way to regulate functional determinants in the evaluation of path integrals. Our proof is essentially the one given by Riemann in his seminal paper.

Before turning to the zeta function we will first prove the functional equation for the Gamma function as a warm-up exercise. In both cases we start from an expression for the respective function that is well-defined only on part of the complex plane. We will then find a certain contour integral that allows an extension of the function to the whole complex plane. Both functional equations follow from certain contour integrals on the complex plane.[1]

Exercise 1: The Gamma function

Before turning to the zeta function we will first derive the Euler reflection formula of the Gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}\,, \tag{2}$$

as a warm-up exercise. The Euler Gamma function is defined by the integral

$$\Gamma(s) = \int_0^\infty dt\, e^{-t} t^{s-1} \qquad \mathrm{Re}(s) > 0\,. \tag{3}$$

1. Use the integral representation (3) of the Gamma function to derive the functional equation

$$\Gamma(s+1) = s\,\Gamma(s)\,. \tag{4}$$

   This shows that we have $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$, so that the Gamma function can be regarded as a generalization of the factorial.

2. Use (4) to argue that the Gamma function can be analytically extended to the strip $-1 < \mathrm{Re}(s) \leq 0$ except at the point $s = 0$.

   Repeat this argument to find that the Gamma function is a meromorphic function on the complex plane with simple poles at $0, -1, -2, ..$ and residues

$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!} \qquad n = 0, 1, 2, \ldots\,. \tag{5}$$

[1] Discussions of the Gamma function and zeta function are obviously ample in the mathematical literature. A nice discussion with a level of rigor appropriate for physicists can be found in the lecture notes of C. Pope: `https://people.tamu.edu/~c-pope/methods.pdf` .

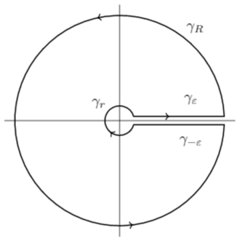

Figure 1: The keyhole contour.

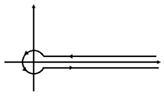

Figure 2: The Hankel contour.

3. We turn now to prove the Euler reflection formula (2). Use the integral representation (3) for each factor to write
$$\Gamma(s)\Gamma(1-s) = \int_0^\infty d\lambda\, \frac{\lambda^{-s}}{1+\lambda}\,, \qquad 0 < \mathrm{Re}(s) < 1\,. \tag{6}$$
Hint: The two integrals on the left hand side can be turned into a double integral. This can be simplified by a smart change of variables that allows to do one of the integrals immediately.

4. Consider now the contour integral
$$\oint_{\mathcal{C}} dz \frac{z^{-s}}{1+z}\,. \tag{7}$$
and let $\mathcal{C}$ denote the keyhole contour in figure 1. Show that the contributions of both circles vanish for $0 < \mathrm{Re}(s) < 1$, and use the residue theorem to show that the integral reduces to
$$\int_0^\infty d\lambda\, \frac{\lambda^{-s}}{1+\lambda} = \frac{\pi}{\sin \pi s}\,. \tag{8}$$
Hint: The function $z^{-s} = e^{-s \ln z}$ is multi-valued. For the keyhole contour it is convenient to put the branch cut along the positive real axis, i.e., $\ln z = \ln|z| + i\theta$ with $0 < \theta < 2\pi$. Then for the path above the branch cut we have $z^{-s} = t^{-s}$ with $t \in (R, r)$ while the path below has $z^{-s} = t^{-s}e^{-2\pi i s}$ with $t \in (r, R)$.

Together with (6) this proves the reflection formula for $0 < \mathrm{Re}(s) < 1$. Nevertheless, both sides are analytic except for the isolated poles at $s \in \mathbb{Z}$, so that we can analytically extend the result to all s thus proving (2).

In the above we analytically continued the Gamma function using the functional equation (4). We can also analytically continue the Gamma function using a contour integral. In order to prepare for the case of the zeta function, we consider this alternative procedure.

Consider the line integral
$$\int_{\mathcal{C}_H} dz e^{-z} z^{s-1}\,, \tag{9}$$
defined on the *Hankel contour* shown in figure 2, i.e., the part of keyhole contour consisting of $\gamma_\epsilon \cup \gamma_r \cup \gamma_{-\epsilon}$. The claim is that this provides an analytically continuation for the Gamma function in the form of
$$\Gamma(s) = -\frac{ie^{-i\pi s}}{2 \sin \pi s} \int_{\mathcal{C}_H} dz\, e^{-z} z^{s-1}\,. \tag{10}$$

Note that the restriction to $\mathrm{Re}(s) > 0$ is not needed any longer since the contour circumvents the branch point $z = 0$ and it is well-defined at the asymptotic ends of the Hankel contour since the integrand vanishes for large z. It remains to show that this definition reduces to (3) when $\mathrm{Re}(s) > 0$.

5. Evaluate the contour integral (9). Show that for $\mathrm{Re}(s) > 0$ the contribution coming from the small circle γ_r vanishes and relation (10) follows upon using the definition (3). Since the expressions coincide for $\mathrm{Re}(s) > 0$, (10) is an analytic continuation for the Gamma function.

Exercise 2: The zeta function

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad \mathrm{Re}(s) > 1\,. \tag{11}$$

We first want to find the analytic extension of this function. In order to do this, it will be convenient to have an integral representation for the zeta function.

1. We start by establishing the following relation:

$$\int_0^{\infty} e^{-nt} t^{s-1} = \Gamma(s) n^{-s} \qquad n \in \mathbb{N}\,. \tag{12}$$

   Use this to obtain the following integral expression for the zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, \frac{t^{s-1}}{e^t - 1} \qquad \mathrm{Re}(s) > 1\,. \tag{13}$$

   Hint: In deriving this, you can assume that you can safely change the order of sum and integral.

The analytic extension of the zeta function follows along the same lines as for the Gamma function in point 5 of the last exercise. Consider the line integral

$$\int_{\mathcal{C}_H} dz\, \frac{z^{s-1}}{e^z - 1} \tag{14}$$

along the Hankel contour. Note that the contour has to circle the origin close enough to avoid the poles at $\pm 2i\pi, \pm 4i\pi, \ldots$. We claim that this provides an analytic continuation for the zeta function in the form of

$$\zeta(s) = \frac{e^{-i\pi s}}{2i\Gamma(s)\sin \pi s} \int_{\mathcal{C}_H} dz\, \frac{z^{s-1}}{e^z - 1} = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1-s) \int_{\mathcal{C}_H} dz\, \frac{z^{s-1}}{e^z - 1}\,, \tag{15}$$

where we used the reflection formula (2) in the last step. The contour integral in (15) is single-valued and analytic for all s since it does not pass through the point $z = 0$. This provides an analytic continuation for the zeta function if we can show that it reduces to (13) in the case $\mathrm{Re}(s) > 1$.

2. Evaluate the line integral (14) and use equation (13) to establish (15) for the case $\mathrm{Re}(s) > 1$.

The only possible non-analyticities of $\zeta(s)$ can thus come from $\Gamma(1-s)$. We saw that this has poles at $s = 1, 2, 3, \ldots$ However, we know from the series definition (11) that $\zeta(s)$ at $s = 2, 3, ..$ is analytic. These poles must therefore cancel against zeros of the contour integral. The only remaining value to check is $s = 1$. For $s = 1$ the contour integral (14) becomes

$$\int_{\mathcal{C}_H} dz\, \frac{1}{e^z - 1}\,. \tag{16}$$

Note that this is a single-valued function and there is now no branch cut anymore along the positive real axis. We can deform the contour by adding a little vertical piece connecting the two end points of the Hankel contour to close the contour. Since the integrand vanishes at the right-hand endpoints of the Hankel contour, we can neglect the contribution of this connecting piece.

3. Evaluate the contour integral (16) around the closed contour surrounding the origin. Conclude that the zeta function behaves as
$$\zeta(s) \sim -\Gamma(1-s) \tag{17}$$
near $s = 1$ and so indeed has a pole at $s = 1$.

Taken together, we analytically continued the zeta function (11) to a meromorphic function on the whole complex plane with a simple pole at $s = 1$. We can now use this integral representation to tease out some values for the zeta function which were previously not defined. For instance:

4. Evaluate the integral representation of the zeta function (15) for $s = 0$. Since there is no branch cut anymore, one can again add a little connecting piece on the end of the Hankel contour and consider a closed contour integral. You should find the result
$$\zeta(0) = -\frac{1}{2}. \tag{18}$$

Going back to the original series definition, we can view this as providing a regularization of the series
$$\zeta(0) = \sum_{n=1}^{\infty} n^0 = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\ldots = -\frac{1}{2}. \tag{19}$$

We will find this regularization of infinite sums due to the zeta function very useful in the computation of determinants later in the course.

We are now finally in the position to prove the reflection formula (1).

5. Consider the integral
$$\int_{\mathcal{C}} dz \, \frac{z^{s-1}}{e^z - 1} \tag{20}$$
where the contour (20) is now the keyhole contour of figure 1. Evaluate this integral using the residue theorem to find
$$\int_{\mathcal{C}} dz \, \frac{z^{s-1}}{e^z - 1} = 2^{s+1}\pi^s i e^{i\pi s} \sin(\pi \frac{s}{2})\zeta(1-s) \qquad \mathrm{Re}(s) < 0\,. \tag{21}$$
Show that the contribution of the circle at infinity vanishes for $\mathrm{Re}(s) < 0$. Conclude that the only contribution to the left hand side of (21) comes from the Hankel contour. We have therefore shown the equation
$$\zeta(s) = 2^s \pi^{s-1}\zeta(1-s)\Gamma(1-s)\sin(\pi\frac{s}{2})\,, \qquad \mathrm{Re}(s) < 0\,. \tag{22}$$

Upon using Euler's reflection formula (2) and the Legendre duplication formula
$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\,\Gamma(2s)\,, \tag{23}$$
we can rewrite (22) in the form
$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\,, \qquad \mathrm{Re}(s) < 0\,. \tag{24}$$

Since both sides are meromorphic functions on the complex plane, this equation can be extended to any s. We have thus proved Riemann's reflection formula.

Solutions to exercise 1

1. From the the defining integral (3) we find

$$\Gamma(s+1)=\int_0^\infty dt\, e^{-t}\, t^s=-e^{-t}t^s|_0^\infty+s\int_0^\infty dte^{-t}t^{s-1}=s\Gamma(s)-e^{-t}t^s|_0^\infty. \tag{25}$$

The last term vanishes for $\mathrm{Re}(s)>0$ and we have proved the identity under this assumption.

2. From (4) we have the relation

$$\Gamma(s)=\frac{\Gamma(s+1)}{s}. \tag{26}$$

Note that the right hand side is analytic and well-defined for $-1<\mathrm{Re}(s)<0$. We have therefore found an analytic extension for the Gamma function to $-1<\mathrm{Re}(s)$.

From this expression we see clearly that there is a pole at $s=0$. The residuum is given by

$$\lim_{s\to 0}s\Gamma(s)=\lim_{s\to 0}\Gamma(s+1)=\Gamma(1)=1. \tag{27}$$

We can continue stripwise and find the analytic extension of the Gamma function to the strip $-(n+1)<\mathrm{Re}(s)<-n$ as given by

$$\Gamma(s)=\frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\dots(s+1)s}. \tag{28}$$

From this we find a pole at $s=-n$ with residuum given by

$$\lim_{s\to -n}(s+n)\Gamma(s)=\lim_{s\to -n}\frac{\Gamma(s+n+1)}{(s+n-1)\dots(s+1)s}=\frac{\Gamma(1)}{(-1)^n n!}=\frac{(-1)^n}{n!}. \tag{29}$$

3. We have

$$\Gamma(s)\Gamma(1-s)=\int_0^\infty dt\int_0^\infty due^{-t}e^{-u}u^{s-1}t^{-s}. \tag{30}$$

We make the following substitution

$$t=xy\, u=(1-x)y\to t+u=y. \tag{31}$$

The integral turns into

$$\Gamma(s)\Gamma(1-s)=\int_0^\infty dy\int_0^1 dxe^{-y}y(1-x)^{s-1}y^{s-1}x^{-s}y^{-s}=\int_0^\infty dye^{-y}\int_0^1 dx\frac{(1-x)^{s-1}}{x^s}. \tag{32}$$

The first integral is trivial and we're left with

$$\Gamma(s)\Gamma(1-s)=\int_0^1 dx\frac{(1-x)^{s-1}}{x^s}. \tag{33}$$

This can be brought into the form above by setting $x/(1-x)=\lambda$. We have then using $d\lambda=dx(1-x)^{-2}$ and $1+\lambda=(1-x)^{-1}$.

$$\Gamma(s)\Gamma(1-s)=\int_0^\infty d\lambda\frac{\lambda^{-s}}{1+\lambda}. \tag{34}$$

4. We will now evaluate this integral using the keyhole contour. We use the following parametrizations for the different paths

$$\gamma_\epsilon: z=t \qquad t\in(r,R) \tag{35}$$
$$\gamma_R: z=Re^{it} \qquad t\in(0,2\pi) \tag{36}$$
$$\gamma_{-\epsilon}: z=te^{2\pi i} \qquad t\in(R,r) \tag{37}$$
$$\gamma_r: z=re^{it} \qquad t\in(2\pi,0). \tag{38}$$

Consider the contributions of paths γ_ϵ and $\gamma_{-\epsilon}$. We have

$$\int_r^R dt \frac{t^{-s}}{1+t} + \int_R^r dt \frac{(te^{2\pi i})^{-s}}{1+te^{2\pi i}} = \int_r^R dt \frac{t^{-s}(1-e^{-2\pi i s})}{1+t}. \tag{39}$$

In the limit $r \to 0$ and $R \to \infty$ this becomes

$$\int_0^\infty dt \frac{t^{-s}}{1+t}(1-e^{-2\pi i s}) = (1-e^{-2\pi i s})\Gamma(s)\Gamma(1-s). \tag{40}$$

For the two circles we have

$$\gamma_R : |\int_0^{2\pi} dt R i e^{it} \frac{R^{-s}e^{-ist}}{1+Re^{it}}| < 2\pi |\frac{R^{1-s}e^{i(1-s)t^*}}{1+Re^{it^*}}|. \tag{41}$$

For $R \to \infty$ this vanishes if s > 0. Similarly, we have the same expression with R replaced by r. We find that this vanishes for $r \to 0$ if $1 - \mathrm{Re}(s) > 0$. We have therefore the equality.

$$\int_{\mathcal{C}} dz \frac{z^{-s}}{1+z} = (1-e^{2\pi i s})\Gamma(s)\Gamma(1-s) \qquad 0 < \mathrm{Re}(s) < 1. \tag{42}$$

On the other hand we can evaluate the contour integral by Cauchy integral formula around the residue $z = -1 = e^{i\pi}$. We find

$$\int_{\mathcal{C}} dz \frac{z^{-s}}{1+z} = 2\pi i e^{-si\pi}. \tag{43}$$

We have therefore shown

$$\Gamma(s)\Gamma(1-s) = \frac{2\pi i e^{-si\pi}}{(1-e^{-2\pi i s})} = \frac{2\pi i}{e^{is\pi}-e^{-is\pi}} = \frac{\pi}{\sin \pi s}. \tag{44}$$

5. Let us evaluate the contour integral in (9). Consider again the paths above and below the branch cut. We have

$$\int_\infty^\epsilon e^{-t}t^{s-1} + \int_\epsilon^\infty e^{-t}t^{s-1}e^{2\pi i(s-1)} = \int_\epsilon^\infty e^{-t}t^{s-1}\Big(e^{2\pi i(s-1)} - 1\Big). \tag{45}$$

For $s > 0$ we can replace the first integral by the definition of the Γ function and find

$$\Gamma(s) = \int_{\mathcal{C}_H} e^{-t}t^{s-1}\frac{1}{\big(e^{2\pi i(s-1)}-1\big)} = \frac{e^{-i\pi s}}{2i\sin \pi s}\int_{\mathcal{C}_H} e^{-t}t^{s-1}. \tag{46}$$

It remains to show that the integral over the small circle vanishes. We have

$$|\int_0^{2\pi} dt i\epsilon e^{it}e^{-\epsilon e^{it}}\epsilon^{s-1}e^{(s-1)it}| < 2\pi|\epsilon^s e^{-\epsilon e^{it^*}}e^{(s-1)it^*}| \to 0 \qquad \mathrm{Re}(s) > 0. \tag{47}$$

Solutions to exercise 2

1. We want to find an integral representation for the zeta function. Using the definition of the Gamma function we start from the identity

$$\int_0^\infty dt e^{-nt}t^{s-1} = \int_0^\infty d(\frac{t}{n})e^{-t}(\frac{t}{n})^{s-1} = n^{-s}\int_0^\infty dt e^{-t}t^{s-1} = n^{-s}\Gamma(s)\,. \tag{48}$$

for any $n \in \mathbb{N}$. Note that we have to assume $\mathrm{Re}(s) > 0$ to use the integral representation of the Gamma function. Next, we can use the defining property of the zeta function to write

$$\sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty dt e^{-nt}t^{s-1} = \int_0^\infty dt \sum_{n=1}^\infty e^{-nt}t^{s-1} = \int_0^\infty dt(\frac{1}{1-e^{-t}}-1)t^{s-1} = \int_0^\infty dt \frac{t^{s-1}}{e^t-1}\,. \tag{49}$$

In deriving this, we switched the order of sum and integral. More rigorously, one would need to show that this step is allowed using e.g. dominated convergence of the sum, but we will forgo this step. Assuming furthermore that $\mathrm{Re}(s) > 1$ so we can use the defining equation of the zeta function, we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t - 1} \,. \tag{50}$$

This is an integral representation for the zeta function valid for $\mathrm{Re}(s) > 1$.

2. As for the Gamma function, we want to find an integral representation valid for any s. Consider therefore the contour integral

$$\int_{C_H} dz \frac{z^{s-1}}{e^z - 1} \tag{51}$$

defined on the Hankel contour. Note that this integral converges due to factor of e^{-z} in the denominator; since the contour evades the branch cut and the potential pole at $z = 0$, this is well-defined for any value of s. We want to show that for $\mathrm{Re}(s) > 1$ this coincides with the above integral representation of the zeta function.

Let us evaluate this integral. We parametrize the three parts of the Hankel contour by dividing it into three paths parametrized as: (i) $z = t, t \in (\infty, \epsilon)$; (ii) $z = \epsilon e^{i\varphi}, \varphi \in (0, 2\pi)$; (iii) $z = e^{2\pi i} t, t \in (\epsilon, \infty)$. We have therefore

$$\int_{C_H} dz \frac{z^{s-1}}{e^z - 1} = \int_\infty^\epsilon dt \frac{t^{s-1}}{e^t - 1} + i \int_0^{2\pi} d\varphi \epsilon e^{i\varphi} \frac{\epsilon^{s-1} e^{i\varphi(s-1)}}{e^{\epsilon e^{i\varphi}} - 1} + \int_\epsilon^\infty dt e^{2\pi i} \frac{t^{s-1} e^{2\pi i(s-1)}}{e^{t e^{2\pi i}} - 1} \tag{52}$$

$$= \int_\epsilon^\infty dt \frac{t^{s-1}}{e^t - 1} (-1 + e^{2\pi i(s-1)}) + i \int_0^{2\pi} d\varphi \frac{\epsilon^s e^{i\varphi s}}{e^{\epsilon e^{i\varphi}} - 1} \,. \tag{53}$$

Let us look at the last contribution. We can bound this as

$$|i \int_0^{2\pi} d\varphi \frac{\epsilon^s e^{i\varphi s}}{e^{\epsilon e^{i\varphi}} - 1}| < 2\pi | \frac{\epsilon^s e^{i\varphi_* s}}{e^{\epsilon e^{i\varphi_*}} - 1} | \tag{54}$$

where φ_* denotes the value of φ_* on which this takes its maximum. Since $\mathrm{Re}(s) > 1$ this goes to zero if ϵ vanishes. We can therefore write in the limit $\epsilon \to 0$

$$\int_{C_H} dz \frac{z^{s-1}}{e^z - 1} = \Gamma(s)\zeta(s)(-1 + e^{2\pi i(s-1)}) = \Gamma(s)\zeta(s)(-e^{-i\pi s} + e^{i\pi s}) e^{i\pi s} \qquad \mathrm{Re}(s) > 1. \tag{55}$$

and therefore

$$\zeta(s) = \frac{e^{-i\pi s}}{2i\Gamma(s) \sin \pi s} \int_{C_H} dz \frac{z^{s-1}}{e^z - 1} = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1 - s) \int_{C_H} dz \frac{z^{s-1}}{e^z - 1} \,, \tag{56}$$

where in the second equation we have used the reflection formula for the Gamma function.

We have shown this for $\mathrm{Re}(s) > 1$. However, the right hand side extends to an analytic function for any s. We have therefore established a good integral representation for the zeta function.

3. As explained in the exercise sheet, the only pole for the zeta function could come from the value $s = 1$ of the above integral

$$\int_{C_H} dz \frac{1}{e^z - 1} \,. \tag{57}$$

Since there is no branch cut, we can deform the contour and add a little connecting piece that we can subsequently push to infinity. This integral is readily evaluated using the series expansion

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \dots \,. \tag{58}$$

The residue theorem yields therefore the value

$$\int_{C_H+conn} dz \frac{1}{e^z-1} = 2\pi i\,. \tag{59}$$

Plugging this back into the above formula (56) we see that the pole of the Gamma function at $s=0$ is not cancelled, and around $s=0$ the zeta function behaves as

$$\zeta(s) \sim -\Gamma(1-s)\,, \tag{60}$$

and therefore has a simple pole of residue 1.

4. We can use the integral representation (56) to evaluate the zeta function for values on which the original series definition was not defined. For instance, for $s=0$ we have

$$\zeta(0) = \frac{1}{2\pi i}\Gamma(1)\int_{C_H} dz \frac{z^{-1}}{e^z-1} = \frac{1}{2\pi i}\int_{C_H+conn} dz \frac{1}{z(e^z-1)} = \frac{1}{2\pi i}\int_{C_H+conn} dz \left(\frac{1}{z^2} - \frac{1}{2z} + \ldots\right) = -\frac{1}{2}\,, \tag{61}$$

where we added again a little connecting piece to close the contour. Similarly, for $s=-1$ we have

$$\zeta(-1) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(2)\int_{C_H} dz \frac{z^{-2}}{e^z-1} = -\frac{1}{2\pi i}\int_{C_H+conn} dz \frac{1}{z^2(e^z-1)} \tag{62}$$

$$= -\frac{1}{2\pi i}\int_{C_H+conn} dz \left(\frac{1}{z^3} - \frac{1}{2z^2} + \frac{1}{12z}\ldots\right) = -\frac{1}{12}\,. \tag{63}$$

5. Now we are finally in the position to prove the Riemann reflection formula. Consider the integral

$$\int_{C_K} dz \frac{z^{s-1}}{e^z-1}\,, \tag{64}$$

which is the same function as above but now evaluated on the keyhole contour. The residue theorem yields the value of this integral. Note that the contour avoids the pole at $z=0$, but there are additional poles at $z=\pm 2\pi i, \pm 4\pi i, \ldots$. If the outer arcs of the keyhole contour are pushed to infinity, we have therefore

$$\int_{C_K} dz \frac{z^{s-1}}{e^z-1} = 2\pi i \sum_{n=1}^{\infty}\left((2n\pi e^{\frac{i\pi}{2}})^{s-1} + (2n\pi e^{\frac{3i\pi}{2}})^{s-1}\right) = 2^s\pi^s i\left(e^{\frac{i\pi}{2}(s-1)} + e^{\frac{3i\pi}{2}(s-1)}\right)\sum_{n=1}^{\infty} n^{1-s} \tag{65}$$

$$= 2^s\pi^s e^{i\pi s}\left(+e^{\frac{-i\pi}{2}s} - e^{\frac{i\pi}{2}s}\right)\sum_{n=1}^{\infty} n^{1-s} = -2^{s+1}\pi^s e^{i\pi s} i \sin\frac{\pi s}{2}\sum_{n=1}^{\infty} n^{1-s}\,. \tag{66}$$

Note that the sum converges for $\mathrm{Re}(s) < 0$.

On the other hand, we can evaluate the keyhole contour by explicitly parametrizing the curves. It then consists of the inverted Hankel contour plus the arcs at infinity. This yields

$$\int_{C_K} dz \frac{z^{s-1}}{e^z-1} = -\int_{C_H} dz \frac{z^{s-1}}{e^z-1} + i\int_0^{2\pi} d\varphi \frac{R^s e^{i\varphi(s)}}{e^{Re^{\varphi i}-1}}\,. \tag{67}$$

Note that the contribution of the arcs at infinity vanishes for $\mathrm{Re}(s) < 0$, and the keyhole contour reduces to the Hankel contour. Equating the two sides of (67) and (65) and plugging in the definition of the zeta function in terms of the Hankel contour yields

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)2^{s+1}\pi^s e^{i\pi s} i \sin\frac{\pi s}{2}\zeta(1-s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)\,. \tag{68}$$

Although we derived this for $\mathrm{Re}(s) < 0$, both sides are analytic in s and this relation remains true for any s. Upon using the Legendre duplication formula for the Gamma function, this becomes the Riemann reflection formula.